Hankel-type determinants and Drinfeld quasi-modular forms*

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In the memory of David Hayes

Abstract. In this paper we introduce a class of determinants "of Hankel type". We use them to compute certain remarkable families of Drinfeld quasi-modular forms.

Contents

1 Introduction

To motivate this paper, we will first review some problem in the classical theory of quasi-modular forms. Let $M_{\mathbb{Z}}$ be the $\mathbb{Z}$-algebra generated by classical modular forms (for $\mathbf{SL}_2(\mathbb{Z})$) whose q-expansion has coefficients in $\mathbb{Z}$. It is well known that $M_{\mathbb{Z}}$ is the polynomial algebra $\mathbb{Z}[E_4, E_6, \Delta]$ where E_4, E_6 are the normalised ([1]) Eisenstein series of weights $4, 6$ respectively, and where $\Delta = (E_4^3 - E_6^2)/1728$ is the unique normalised cusp form of weight 12, so that, in particular, $M_{\mathbb{Z}}$ is finitely generated.

Let now $\widetilde{M}_{\mathbb{Q}}$ be the $\mathbb{Q}$-algebra of classical quasi-modular forms, as defined by Kaneko and Zagier in [6], with the additional condition that their q-expansions have coefficients in $\mathbb{Q}$. It is easy to show that $\widetilde{M}_{\mathbb{Q}} = \mathbb{Q}[E_2, E_4, E_6]$, where E_2 is the (non-modular) normalised Eisenstein series of weight 2, so this again is a finitely generated algebra, but over $\mathbb{Q}$.

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[1] A formal power series $\sum_{i\geq i_0} c_i q^i$ (or $\sum_{i\geq i_0} c_i u^i$) is said to be normalised if $c_{i_0} = 1$. A modular form is normalised, by definition, if its q-expansion is normalised. A similar definition will be used for quasi-modular forms and for Drinfeld quasi-modular forms.

We may then formulate the following:

Problem 1. *Compute a minimal set of generators for $\widetilde{M_{\mathbb{Z}}} = \widetilde{M_{\mathbb{Q}}} \cap \mathbb{Z}[[q]]$, the $\mathbb{Z}$-algebra generated by quasi-modular forms of $\widetilde{M_{\mathbb{Q}}}$ whose q-expansions have coefficients in $\mathbb{Z}$.*

This problem is likely to be a difficult one. The examination of the q-expansions of the quasi-modular forms DE_w with E_w normalised Eisenstein series of weight w, $D = qd/dq$ and Clausen-von Staudt Theorem, indicate that the algebra $\widetilde{M_{\mathbb{Z}}}$ is more likely not finitely generated, in contrast with the structure of $M_{\mathbb{Z}}$. What does a minimal set of generators of $\widetilde{M_{\mathbb{Z}}}$ look like?

In [7], Kaneko and Koike introduced a notion of *extremal quasi-modular form* ([2]). An extremal quasi-modular form of weight w and depth $\leq l$ is a non-vanishing polynomial in E_2, E_4, E_6 which is isobaric of weight w, whose degree in E_2 is not bigger than l, and such that the order of vanishing at $q = 0$ of its q-expansion is maximal. If $w \geq 0$ is even and $l \geq 0$, such a form exists and is proportional to a unique normalised form in $\widetilde{M_{\mathbb{Q}}}$ denoted by $f_{l,w}$.

Kaneko and Koike, in [7, Conjecture 2], made a prediction on the size of the denominators of the coefficients of such forms which resembles in some way to a generalisation of Clausen-von Staudt Theorem. Indeed, if Conjecture 2 of loc. cit. holds, then $f_{l,w} \in \mathbb{Z}_p[[q]]$ for every prime number p such that $p \geq w$, *provided* that $l \leq 4$. In addition, the following question can be addressed.

Question. *Let l be a non-negative integer, and denote by $\mathcal{E}_l$ the set of w's such that $f_{l,w}$ exists, and belongs to $\mathbb{Z}[[q]]$. For which l's is $\mathcal{E}_l$ infinite?*

Although very few of the $f_{l,w}$'s are known to have q-expansion defined over $\mathbb{Z}$ ([3]), the feeling that we have, after extensive numerical computations, is that $\mathcal{E}_l$ is infinite for $0 \leq l \leq 4$ and finite for $l > 4$. In these circumstances, we would suggest to use these forms $f_{l,w}$ with w in $\mathcal{E}_l$ to construct a set of generators for $\widetilde{M_{\mathbb{Z}}}$ but we refrain from making any kind of written prediction in this direction because this hypothesis is, so far, largely conjectural.

In this paper, we want to discuss similar problems, arising in the theory of *Drinfeld quasi-modular forms,* where we have a slightly better understanding of what is going on. Let $q = p^e$ be a power of a prime number p with $e > 0$ an integer, let $\mathbb{F}_q$ be the finite field with q elements. Let us consider, for an indeterminate θ, the polynomial ring $A = \mathbb{F}_q[\theta]$ and its fraction field $K = \mathbb{F}_q(\theta)$.

Let K_∞ be the completion of K for the θ^{-1}-adic valuation and let us embed an algebraic closure of K_∞ in its completion $\mathbb{C}_\infty$ for the unique extension of that valuation. Following Gekeler in [5], we denote by Ω the set $\mathbb{C}_\infty \setminus K_\infty$, which has a structure of a rigid analytic space over which the group $\Gamma = \mathbf{GL}_2(A)$ acts discontinuously by homographies, and with the usual local parameter at infinity u (denoted by t in [5] and [2]). These facts lead quite naturally to the notion of *Drinfeld quasi-modular forms*, rather parallel to that of classical quasi-modular forms for $\mathbf{SL}_2(\mathbb{Z})$, which are studied in [2], and to which we refer for the required background.

Following [2], we have three remarkable formal series $E, g, h \in A[[u]]$ algebraically independent over $K(u)$, representing respectively: the u-expansion of a Drinfeld quasi-modular form of weight 2, type 1 and depth 1 (the false Eisenstein series of weight 2 of Gekeler [5]), the u-expansion of an Eisenstein series of weight $q-1$ and type 0, and the u-expansion of a Poincaré series of weight $q+1$ and type 1. The first terms of these formal series are as follows, where $[i] = \theta^{q^i} - \theta$ for $i > 0$ integer

[2]Notice that in fact, the definition of extremality of Kaneko and Koike slightly differs from ours.
[3]For example, it is not known whether $f_{1,14} \in \mathbb{Z}[[q]]$ but this looks true from numerical evidence.

(see [2, Lemma 4.2]):

$$\begin{aligned} E &= u + u^{q^2-2q+2} + \cdots \in uA[[u^{q-1}]] \\ g &= 1 - [1]u^{q-1} - [1]u^{q^3-2q^2+2q-1} + \cdots \in A[[u^{q-1}]] \\ h &= -u - u^{q^2-2q+2} + \cdots \in uA[[u^{q-1}]]. \end{aligned}$$

Let $M_{w,m}$ be the K-vector space of *Drinfeld modular forms of weight w, type m, whose u-expansions are defined over K*, which also is the space of isobaric polynomials (for weights and types) in g and h with coefficients in K ([4]). The K-vector space of *Drinfeld quasi-modular forms of weight w, type m and depth $\leq l$, defined over K* is the space

$$\widetilde{M}^{\leq l}_{w,m} = M_{w,m} \oplus M_{w-2,m-1}E \oplus \cdots \oplus M_{w-2l,m-l}E^l.$$

All these spaces are finite dimensional subspaces of $K[[u]]$ and we may form the K-algebra of Drinfeld quasi-modular forms

$$\widetilde{M}_K = K[E,g,h] = \bigoplus_{w,m} \bigcup_l \widetilde{M}^{\leq l}_{w,m}.$$

In analogy with the Problem 1, we have:

Problem 2. *Compute a minimal set of generators for $\widetilde{M}_A$, the A-algebra generated by quasi-modular forms of $\widetilde{M}_K$ whose u-expansions have coefficients in A.*

We say that an element f of $\widetilde{M}^{\leq l}_{w,m} \setminus \{0\}$ is an *extremal Drinfeld quasi-modular form* if $\mathrm{ord}_{u=0} f$ is maximal among the orders at $u=0$ of non-zero elements of that vector space. If there exists an extremal Drinfeld quasi-modular form of $\widetilde{M}^{\leq l}_{w,m}$ ([5]), we denote by $f_{l,w,m}$ the unique normalised such form.

To present our main result, we need to define a certain double sequence of quasi-modular forms

$$(E_{j,k})_{j\in\mathbb{Z},k\geq 1}.$$

Let us denote as usual by $\Delta = -h^{q-1} \in A[[u]]$ the opposite of the unique normalised cusp form of weight q^2-1 for $\mathbf{GL}_2(A)$, and let us extend the notation $[j]$ to non-positive integers by simply writing $[j] = \theta^{q^j} - \theta$ for $j \in \mathbb{Z}$, so that $[0] = 0$ and $[-1] = \theta^{1/q} - \theta$. The sub-sequence $(E_{j,1})_{j\in\mathbb{Z}}$ is defined inductively in the following way. We set $E_{0,1} = E$, $E_{1,1} = -\frac{Eg+h}{[1]}$ and then, for $j \geq 0$, by

$$E_{j+2,1} = -\frac{1}{[j+2]}(\Delta^{q^j} E_{j,1} + g^{q^{j+1}} E_{j+1,1}),$$

and for $j \leq 1$, by

$$E_{j-2,1} = -\frac{1}{\Delta^{q^{j-2}}}([j]E_{j,1} + g^{q^{j-1}} E_{j-1,1}).$$

[4]Properly speaking, to call these spaces "spaces of Drinfeld modular forms" is an abuse of language; these spaces are just generated by the u-expansions associated to such forms, but since we will work here with formal series in u only, it looked advantageous to make the identification between forms and formal series. We will do the same for Drinfeld quasi-modular forms; see [2] for further explanations.

[5]This occurs if and only if $\widetilde{M}^{\leq l}_{w,m} \neq (0)$, that is, if and only if $w \equiv 2m \pmod{q-1}$ with $w, l \geq 0$), it is unique up to multiplication by an element of $K^\times := K \setminus \{0\}$.

For example, we have the following particular cases:

$$\begin{aligned}
E_{-1,1}^{q} &= -h, \\
E_{-2,1}^{q^2} &= -hg^q, \\
E_{-3,1}^{q^3} &= -h(g^{q+1}-[1]^q h^{q-1})^q,
\end{aligned}$$

and in general, for all $j \leq -1$, it is possible to check that $E_{-j,1}^{q^j}$ is a Drinfeld cusp form of weight q^j+1 and type 1.

Let us write

$$B_k(t) := \prod_{0\leq i<j<k} (t^{q^j}-t^{q^i}) \in \mathbb{F}_q[t].$$

For $k \geq 2$ and $j \in \mathbb{Z}$, we then define $E_{j,k}$ with the following *determinant of Hankel type*:

$$E_{j,k} = \frac{1}{B_k(\theta)} \begin{vmatrix} E_{j,1} & E_{j+1,1} & \cdots & E_{j+k-1,1} \\ E_{j-1,1}^{q} & E_{j,1}^{q} & \cdots & E_{j+k-2,1}^{q} \\ E_{j-2,1}^{q^2} & E_{j-1,1}^{q^2} & \cdots & E_{j+k-3,1}^{q^2} \\ \vdots & \vdots & & \vdots \\ E_{j-k+1,1}^{q^{k-1}} & E_{j-k+2,1}^{q^{k-1}} & \cdots & E_{j,1}^{q^{k-1}} \end{vmatrix}.$$

We shall show:

Theorem 1 *The following properties hold, for $j \geq 0$ and $k \geq 1$.*

1. *There exists a constant $C(q,k)$ and a sequence of integers $(l_k)_{k\geq 1}$ such that for all $j \geq C(q,k)$,*

$$E_{j,k} \in \widetilde{M}^{\leq (q^k-1)/(q-1)}_{(q^k-1)(q^j+1)/(q-1),k} \setminus \widetilde{M}^{\leq l_k}_{(q^k-1)(q^j+1)/(q-1),k}$$

   with $l_k \to \infty$ for $k \to \infty$.

2. *For all j,k with $j \geq 0$, we have $\mathrm{ord}_{u=0} E_{j,k} = q^j(q^{2k}-1)/(q^2-1)$.*

3. *For all j,k with $j \geq 0$, we have $E_{j,k} \in A[[u]]$ and $E_{j,k}$ is normalised.*

4. *For $k=1$ and for $k=2$ if $q \geq 3$, we have $E_{j,k} = f_{(q^k-1)/(q-1),(q^k-1)(q^j+1)/(q-1),k}$ for all $j \geq 0$.*

In particular, for all j,k, $E_{j,k}$ is non-zero, property which does not seem to follow directly from the definition above. The interest of the theorem is that it provides in an explicit way a family of *normalised* Drinfeld quasi-modular forms parametrised by $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{>0}$, with *unbounded depths and weights*, with *high order of vanishing* at $u=0$, and with u-expansions *defined over A*. The theorem gives a partial answer to the analogue of the Question above. Indeed, denoting by $\mathcal{E}_{l,m}$ the set whose elements are the weights w such that $f_{l,w,m}$ is defined over A, we have the following obvious consequence of Theorem 1.

Corollary 2 *If $l=1$ and for any value of q, or if $q \geq 3$ and $l = q+1$, we have that $f_{l,l(q^j+1),l} \in A[[u]]$ for all $j \geq 0$. Therefore, for the selected values of q,l,m, the set $\mathcal{E}_{l,m}$ has infinitely many elements.*

It can be shown that for $k>2$, the degree of $E_{j,k}$ in E is not equal to $(q^k-1)/(q-1)$, that is, it is not maximal (it is maximal only for $k=1,2$), which may mean that for such values, $E_{j,k}$ is not extremal. However, the fact that $l_k\to\infty$ suggests that no natural threshold for the depth (as $l=4$ in the classical case, as suggested by [7, Conjecture 2]) exists in the Drinfeldian framework. Moreover, the presence of infinitely many $f_{l,w,m}$'s defined over A detected by Theorem 1 suggests that the A-algebra $\widetilde{M}_A$ generated by the Drinfeld quasi-modular forms with u-expansions defined over A could have, as a minimal set of generators, the $f_{l,w,m}$'s with $w\in\mathcal{E}_{l,m}$ for all l,m's.

Remark. With the help of a formula appearing in [11], it is possible to explicitly compute the u-expansions of $E_{-j,1}^{q^j}$ for $j\geq 0$: we have $E_{-j,1}^{q^j}=\sum_{a\in A^+}a^{q^j}u_a$ with the notations of loc. cit. These forms, which are Hecke eigenforms, are also object of investigations by A. Petrov (private communication).

2 Determinants of Hankel's type

An *inversive difference field* $(\mathcal{K},\tau)$ is the datum of a field $\mathcal{K}$ together with an automorphism τ that will be supposed of infinite order. The *τ-constant subfield* $\mathcal{K}^\tau$ is by definition the subfield of $\mathcal{K}$ of all the elements $x\in\mathcal{K}$ such that $\tau x=x$. Every inversive difference field can be embedded in an *existentially closed* field $\mathcal{K}^{\mathrm{ex}}$, that is a field endowed with an extension of τ such that $\mathcal{K}^\tau=(\mathcal{K}^{\mathrm{ex}})^\tau$, in which every polynomial τ-difference equation has at least a non-trivial solution.

We need now to choose a field $\mathcal{K}$ with *two* distinguished automorphisms to serve our purposes. Consider two indeterminates t,u and the field of formal series

$$\mathcal{R}=K((t))((u)).$$

The Frobenius $\mathbb{F}_q$-linear endomorphism F of $\mathcal{R}$ splits as a product

$$F=\chi\tau=\tau\chi,$$

where $\chi,\tau:\mathcal{R}\to\mathcal{R}$ are respectively $K((u))$- and $\mathbb{F}_q((t))$-linear, uniquely determined by $\chi(t)=t^q$, $\tau(u)=u^q$ and $\tau\theta=\theta^q$. The perfection

$$\mathcal{K}=\mathcal{R}^{\mathrm{perf}}=\bigcup_{i\geq 0}\mathbb{F}_q(\theta^{1/q^i})((t^{1/q^i}))((u^{1/q^i}))$$

of $\mathcal{R}$ is then endowed with extensions of τ and χ such that both the difference fields $(\mathcal{K},\tau)$ and $(\mathcal{K},\chi)$ are inversive. Also, $\mathcal{K}^\tau$ is equal to the perfect closure $\mathbb{F}_q((t))^{\mathrm{perf}}$ of $\mathbb{F}_q((t))$ in $\mathcal{K}$ and $\mathcal{K}^\chi$ is equal to the perfect closure $K((u))^{\mathrm{perf}}$ of $K((u))$ in $\mathcal{K}$.

Let $x_1,\ldots,x_s$ be elements of $\mathcal{K}$. Their *τ-wronskian* is the determinant:

$$W_\tau(x_1,\ldots,x_s)=\det\begin{pmatrix}x_1 & \tau x_1 & \cdots & \tau^{s-1}x_1\\ x_2 & \tau x_2 & \cdots & \tau^{s-1}x_2\\ \vdots & \vdots & & \vdots\\ x_s & \tau x_s & \cdots & \tau^{s-1}x_s\end{pmatrix}.$$

We recall from [10] that $x_1,\ldots,x_s$ are $\mathcal{K}^\tau$-linearly independent if and only if $W_\tau(x_1,\ldots,x_s)\neq 0$. Similarly, the *$\chi$-wronskian* $W_\chi(x_1,\ldots,x_s)$ of $x_1,\ldots,x_s$ can be introduced, and $x_1,\ldots,x_s$ are $\mathcal{K}^\chi$-linearly independent if and only if $W_\chi(x_1,\ldots,x_s)\neq 0$.

For $\boldsymbol{f} \in \mathcal{K}$, we introduce the following sequence of determinants of *Hankel type*:

$$H_k(\boldsymbol{f}) = \begin{vmatrix} \boldsymbol{f} & \tau\boldsymbol{f} & \tau^2\boldsymbol{f} & \cdots & \tau^{k-1}\boldsymbol{f} \\ \chi\boldsymbol{f} & \chi\tau\boldsymbol{f} & \chi\tau^2\boldsymbol{f} & \cdots & \chi\tau^{k-1}\boldsymbol{f} \\ \vdots & \vdots & \vdots & & \vdots \\ \chi^{k-1}\boldsymbol{f} & \chi^{k-1}\tau\boldsymbol{f} & \chi^{k-1}\tau^2\boldsymbol{f} & \cdots & \chi^{k-1}\tau^{k-1}\boldsymbol{f} \end{vmatrix}.$$

The proposition below will be used later.

Proposition 3 *The following conditions are equivalent.*

(i) $H_k(\boldsymbol{f}) = 0$ *for some* $k \geq 1$.

(ii) There exist $s \geq 1$, *elements* $\lambda_1, \ldots, \lambda_s$ *in* $\mathbb{F}_q((t))^{perf}$ *and elements* $b_1, \ldots, b_s$ *in some algebraic closure* $K((u))^{alg}$ *of* $K((u))$ *such that, in some existentially closed extension of* $(\mathcal{K}, \tau)$ *containing* $K((u))^{alg}$,

$$\boldsymbol{f} = \lambda_1 b_1 + \cdots + \lambda_s b_s.$$

(iii) For some $s \geq 1$, *there exist elements* $\mu_1, \ldots, \mu_s$ *in* $K((u))^{perf}$, *and elements* $b'_1, \ldots, b'_s$ *in* $\mathbb{F}_q((t))^{alg}$, *an algebraic closure of* $\mathbb{F}_q((t))$, *such that*

$$\boldsymbol{f} = \mu_1 b'_1 + \cdots + \mu_s b'_s,$$

in some existentially closed extension of $(\mathcal{K}, \chi)$ *containing* $\mathbb{F}_q((t))^{alg}$.

Proof. It is easy to show that each of the second and the third conditions separately implies the first. Let us show that the first condition implies the second. Assuming that $H_k(\boldsymbol{f}) = 0$ for some $k \geq 1$ is equivalent to say that $W_\chi(\boldsymbol{f}, \tau\boldsymbol{f}, \ldots, \tau^{k-1}\boldsymbol{f}) = 0$. Hence, there exist $a_0, \ldots, a_s \in \mathcal{K}^\chi = K((u))^{\mathrm{perf}}$ with $a_0 a_s \neq 0$, such that

$$a_0\boldsymbol{f} + a_1\tau\boldsymbol{f} + \cdots + a_s\tau^s\boldsymbol{f} = 0.$$

On the other hand, the algebraic equation

$$a_0 X + a_1 X^q + \cdots + a_s X^{q^s} = 0$$

has s solutions $b_1, \ldots, b_s$ in an algebraic closure $K((u))^{\mathrm{alg}}$ of $K((u))$, which are linearly independent over the field $(\mathcal{K}^\tau)^F = \mathbb{F}_q$. In particular, $W_F(b_1, \ldots, b_s) \neq 0$.

Let us consider the compositum $\mathcal{F}$ of $\mathcal{K}$ and $K((u))^{\mathrm{alg}}$ in some existentially closed extension of the difference field $(\mathcal{K}, \tau)$ (so we embed $K((u))^{\mathrm{alg}}$ in the existentially closed difference field $(\mathcal{K}^{\mathrm{ex}}, \tau)$). The restriction $\tau|_{K((u))^{\mathrm{alg}}}$ of τ is equal to the restriction of the Frobenius $F|_{K((u))^{\mathrm{alg}}}$. Moreover, obviously, $W_F(b_1, \ldots, b_s) = W_\tau(b_1, \ldots, b_s)$ so that $b_1, \ldots, b_s$ are also $\mathcal{F}^\tau$-linearly independent, $\mathcal{F}^\tau$ being equal to $\mathbb{F}_q((t))^{\mathrm{perf}}$. Since $b_1, \ldots, b_s$ span the $\mathcal{F}^\tau$-vector space of solutions of the equation

$$a_0 X + a_1\tau X + \cdots + a_s\tau^s X = 0,$$

we obtain the second property.

The proof that the first property implies the third is similar and left to the reader, who will notice that it suffices to transpose the matrix used to define $H_k(\boldsymbol{f})$. $\square$

Remark. It is easy to show, writing $H_{s,k}$ at the place of $\tau^s H_k(\boldsymbol{f})$ for a better display, that the following formula holds:

$$H_{s,k}^{q+1} - H_{s,k-1}^q H_{s,k+1} = H_{s-1,k}^q H_{s+1,k}, \quad (s \in \mathbb{Z}, k \geq 2). \tag{1}$$

Formula (1) plays a role for (τ, χ)-difference fields similar to that of Sylvester's formula expressing determinants $\left|\left(\frac{\partial^{i+j} f}{\partial z_1^i \partial z_2^j}\right)_{0 \leq i,j \leq k-1}\right|$ as in [1].

The elements $\boldsymbol{f} = \sum_{i,j} c_{i,j} t^i u^j$ that we choose are either $\boldsymbol{d}$, either $\boldsymbol{E} = -h\tau\boldsymbol{d}$, where $\boldsymbol{d}$ is the unique solution (cf. [9]) in $\mathbb{F}_q[t,\theta][[u]] \subset A[[t]][[u]]$ of the linear τ-difference equation

$$(t - \theta^q)\Delta(\tau^2 X) + g(\tau X) - X = 0, \tag{2}$$

with $c_{0,0} = 1$ and $c_{i,0} = 0$ for $i > 0$. We point out that in [9] we have computed some coefficients of the u-expansion of $\boldsymbol{d}$. See also the remark after Lemma 9 below.

The relationship between $H_k(\boldsymbol{d})$ and $H_k(\boldsymbol{E})$ is simple. Since $\chi h = h$, we have

$$\chi^{i-1}\tau^{j-1}(\boldsymbol{E}) = -h^{q^{j-1}}\tau\big(\chi^{i-1}\tau^{j-1}(\boldsymbol{d})\big) \qquad (1 \leq i,j \leq k),$$

hence

$$H_k(\boldsymbol{E}) = (-1)^k h^{1+\cdots+q^{k-1}}\tau(H_k(\boldsymbol{d})) = (-1)^k h^{\frac{q^k-1}{q-1}}\tau(H_k(\boldsymbol{d})). \tag{3}$$

Lemma 4 *We have, for $j \in \mathbb{Z}$ and $k \geq 1$:*

$$E_{j,k} = \left.\frac{\tau^j H_k(\boldsymbol{E})}{B_k}\right|_{t=\theta}.$$

Proof. For all k, $H_k(\boldsymbol{f})$ can be rewritten, thanks to the identity $\chi = F\tau^{-1}$, as

$$H_k(\boldsymbol{f}) = \begin{vmatrix} \boldsymbol{f} & \tau\boldsymbol{f} & \tau^2\boldsymbol{f} & \cdots & \tau^{k-1}\boldsymbol{f} \\ (\tau^{-1}\boldsymbol{f})^q & \boldsymbol{f}^q & (\tau\boldsymbol{f})^q & \cdots & (\tau^{k-2}\boldsymbol{f})^q \\ \vdots & \vdots & \vdots & & \vdots \\ (\tau^{1-k}\boldsymbol{f})^{q^{k-1}} & (\tau^{2-k}\boldsymbol{f})^{q^{k-1}} & (\tau^{3-k}\boldsymbol{f})^{q^{k-1}} & \cdots & \boldsymbol{f}^{q^{k-1}} \end{vmatrix}. \tag{4}$$

It is proved in [9] that $\boldsymbol{E}|_{t=\theta} = E = E_{0,1}$. Moreover, by Lemma 22 of [9] we have $\tau\boldsymbol{E} = \frac{1}{t-\theta^q}(g\boldsymbol{E}+\boldsymbol{h})$, hence $(\tau\boldsymbol{E})|_{t=\theta} = E_{1,1}$. Now, as one sees from Equation (2) above, or by Proposition 9 of [9], the function $\boldsymbol{E}$ satisfies the linear τ-difference equation

$$(t - \theta^{q^2})(\tau^2\boldsymbol{E}) = g^q(\tau\boldsymbol{E}) + \Delta\boldsymbol{E}.$$

It easily follows from this, by induction, that $(\tau^j\boldsymbol{E})|_{t=\theta}$ is well defined for all $j \in \mathbb{Z}$, and is equal to $E_{j,1}$. Comparing the definition of $E_{j,k}$ with (4) we immediately recover that the forms $E_{j,k}$ of Theorem 1 are, for $k \geq 1$, precisely the formal series of $K[[u]]$ obtained by substituting t with θ in $\tau^j H_k(\boldsymbol{E})/B_k$, a licit operation. □

3 Properties of the determinants $H_k(\boldsymbol{d})$

Let $k \geq 1$ be an integer. Either $H_k(\boldsymbol{d}) = 0$, or there exists $\nu_k \in \mathbb{Z}_{\geq 0}$ such that

$$H_k(\boldsymbol{d}) = \sum_{s \geq \nu_k} \kappa_{k,s} u^s$$

with $\kappa_{k,s} \in \mathbb{F}_q[t,\theta]$ and $\kappa_{k,\nu_k} \neq 0$. We will prove the Theorem below, from which we will deduce Theorem 1.

Theorem 5 *We have $H_k(\boldsymbol{d}) \neq 0$ for all $k \geq 1$, and the following properties hold.*

1. $\nu_k = \frac{(q^k-1)(q^{k-1}-1)}{q^2-1}$,
2. $\kappa_{k,\nu_k} = B_k(t)$,
3. $H_k(\boldsymbol{d})/\kappa_{k,\nu_k}$ *lies in* $\mathbb{F}_q[t,\theta][[u^{q-1}]]$ *and is normalised.*

Corollary 6 *The function $\boldsymbol{d}$ can be expressed neither as a finite linear combination $\lambda_1 b_1 + \cdots + \lambda_s b_s$ with $\lambda_1, \ldots, \lambda_s \in \mathbb{F}_q((t))^{perf}$ and $b_1, \ldots, b_s \in K((u))^{alg}$, nor as a finite linear combination $\mu_1 b_1' + \cdots + \mu_s b_s'$ with $\mu_1, \ldots, \mu_s \in K((u))^{perf}$ and $b_1', \ldots, b_s' \in \mathbb{F}_q((t))^{alg}$.*

Proof. By Theorem 5, $H_k(\boldsymbol{d}) \neq 0$ for all k. Therefore, we can apply Proposition 3. □

The rest of this section is devoted to the proof of Theorem 5. Since the u-expansions of many forms involved (like $g, \Delta, \boldsymbol{d} \ldots$), are actually expansions in powers of u^{q-1}, it will be convenient to set

$$v := u^{q-1}.$$

In Section 3.1, we first prove a general divisibility property for the coefficients of the u-expansion of $H_k(f)$ for formal series $f \in \mathbb{F}_q[t,\theta][[v]]$. Then, in Section 3.2, we carefully study the growth of the degree in t of the coefficients of $\boldsymbol{d}$. Finally, we complete the proof of Theorem 5 in Section 3.3.

3.1 Computation of normalisation factors

Proposition 7 *Let f be a formal series in $\mathbb{F}_q[t,\theta][[v]]$, so that we have a formal series expansion $H_k(f) = \sum_{s \geq 0} \kappa_s v^s$ with $\kappa_s \in \mathbb{F}_q[t,\theta]$ for all s. Then, the polynomial $B_k(t)$ divides κ_s for all $s \geq 0$.*

Proof. We observe that if for $1 \leq i,j \leq k$ we have formal expressions $f_{i,j} = \sum_{s \in \mathcal{I}} c_{i,j,s}$, then, by multilinearity:

$$\begin{vmatrix} f_{1,1} & \cdots & f_{1,k} \\ \vdots & & \vdots \\ f_{k,1} & \cdots & f_{k,k} \end{vmatrix} = \sum_{s_1, \ldots, s_k \in \mathcal{I}} \begin{vmatrix} c_{1,1,s_1} & \cdots & c_{1,k,s_k} \\ \vdots & & \vdots \\ c_{k,1,s_1} & \cdots & c_{k,k,s_k} \end{vmatrix}. \tag{5}$$

Let us write $f = \sum_{s \geq 0} c_s v^s$ with $c_s \in \mathbb{F}_q[t,\theta]$. We set

$$f_{i,j} = \chi^{i-1}\tau^{j-1}(f) = \sum_{s \geq 0} \chi^{i-1}\tau^{j-1}(c_s v^s) = \sum_{s \geq 0} c_s(t^{q^{i-1}}, \theta^{q^{j-1}}) u^{q^{j-1}s}$$

so that $c_{i,j,s}=c_s(t^{q^{i-1}},\theta^{q^{j-1}})v^{sq^{j-1}}$. By (5), we obtain that

$$H_k(f)=\sum_{s_1,s_2,\ldots,s_k}v^{s_1+s_2q+\cdots+s_kq^{k-1}}d_{s_1,s_2,\ldots,s_k}, \tag{6}$$

where

$$d_{s_1,s_2,\ldots,s_k}=\left|\begin{array}{llll} c_{s_1}(t,\theta) & c_{s_2}(t,\theta^q) & \cdots & c_{s_k}(t,\theta^{q^{k-1}}) \\ c_{s_1}(t^q,\theta) & c_{s_2}(t^q,\theta^q) & \cdots & c_{s_k}(t^q,\theta^{q^{k-1}}) \\ \vdots & \vdots & & \vdots \\ c_{s_1}(t^{q^{k-1}},\theta) & c_{s_2}(t^{q^{k-1}},\theta^q) & \cdots & c_{s_k}(t^{q^{k-1}},\theta^{q^{k-1}}) \end{array}\right|. \tag{7}$$

We use the fact that $c_s=\sum_\mu \kappa_{\mu,s}\theta^\mu\in\mathbb{F}_q[t,\theta]$, with $\kappa_{\mu,s}\in\mathbb{F}_q[t]$. Let us apply (5) again, this time with $f_{i,j}=\chi^{i-1}\tau^{j-1}c_{s_j}=\sum_\mu(\chi^{i-1}\kappa_{\mu,s_j})\theta^{\mu q^{j-1}}$ and $c_{i,j,\mu}=(\chi^{i-1}\kappa_{\mu,s_j})\theta^{\mu q^{j-1}}$.

We obtain that

$$d_{s_1,\ldots,s_k}=\sum_{\mu_1,\ldots,\mu_k}\theta^{\mu_1+\mu_2q+\cdots+\mu_{k-1}q^{k-1}}e_{\mu_1,\ldots,\mu_k},$$

where

$$e_{\mu_1,\ldots,\mu_k}=\left|\begin{array}{ccc} \eta_1 & \cdots & \eta_k \\ \vdots & & \vdots \\ \chi^{k-1}\eta_1 & \cdots & \chi^{k-1}\eta_k \end{array}\right|,$$

with $\eta_j=\kappa_{\mu_j,s_j}$. Now, by multilinearity, $e_{\mu_1,\ldots,\mu_k}$ is a sum of Moore's determinants:

$$M(\nu_1,\ldots,\nu_k)=\left|\begin{array}{ccc} t^{\nu_1} & \cdots & t^{\nu_k} \\ \vdots & & \vdots \\ t^{\nu_1q^{k-1}} & \cdots & t^{\nu_kq^{k-1}} \end{array}\right|.$$

We then apply the following lemma, which completes the proof of Proposition 7.

Lemma 8 *The formula*

$$M(0,1,\ldots,k-1)=B_k(t)$$

holds. Moreover, for any choice of $\nu_1,\ldots,\nu_k$, $B_k(t)$ divides $M(\nu_1,\ldots,\nu_k)$.

Proof. The explicit formula is a well known application, either of Moore's determinants, or Vandermonde's determinants. As for the divisibility property, this follows from an old and well known result of Mitchell, [8], as $M(\nu_1,\ldots,\nu_k)$ can be viewed as a generalised Vandermonde's determinant. □

3.2 The degrees of the coefficients of $\boldsymbol{d}$

To prove Theorem 5, we will need a precise estimate of the growth of the degree in t of the coefficients of $\boldsymbol{d}$. Recall that the function $\boldsymbol{d}$ lies in $\mathbb{F}_q[t,\theta][[v]]$, where $v=u^{q-1}$. We will write in what follows

$$\boldsymbol{d}=\sum_{s\geq 0}c_sv^s, \tag{8}$$

where $c_s\in A[t]$. The aim of this section is to prove the following lemma.

Lemma 9 *Let $s \geq 0$ and $l \geq 0$ be integers satisfying*

$$s < 1 + q^2 + \cdots + q^{2l}.$$

Then

$$\deg_t c_s \leq l.$$

Moreover, for all $l \geq 0$ we have

$$c_{1+q^2+\cdots+q^{2(l-1)}}(t) = (-1)^l t^l + \cdots , \tag{9}$$

where the dots stand for terms of degree $< l$.

Remark. We have used here the convention that the empty sum is zero, so we have $1 + q^2 + \cdots + q^{2(l-1)} = 0$ when $l = 0$.
Proof. Write

$$g = 1 - [1]v + \cdots = \sum_{s \geq 0} \gamma_s v^s \in A[[v]],$$

and

$$\Delta = -v(1 - v^{q-1} + \cdots) = \sum_{s \geq 0} \delta_s v^s \in vA[[v]].$$

As in [9], we will use the following recursion formula for the coefficients c_s, which easily follows from the τ-difference equation (2) (see [9, Formula (30)]) :

$$c_s = \sum_{i+jq=s} \gamma_i(\tau c_j) \ + (t - \theta^q) \sum_{i+jq^2=s} \delta_i(\tau^2 c_j). \tag{10}$$

We first prove by induction on $s \geq 0$ that $\deg_t c_s \leq l$ for all l satisfying $1 + q^2 + \cdots + q^{2l} > s$. This statement is clearly true for $s = 0$ and $s = 1$, since $c_0 = 1$ and $c_1 = -(t - \theta)$. Let now $s \geq 2$ and $l \geq 0$ be such that $s < 1 + q^2 + \cdots + q^{2l}$, and consider the formula (10). If j is an index occurring in the first sum, then we have $j \leq s/q < s$, hence

$$\deg_t \tau c_j = \deg_t c_j \leq l \tag{11}$$

by induction hypothesis. Let now (i, j) be a pair of indices occurring in the second sum. If $i = 0$, then $\delta_i = 0$ and $\delta_i(\tau^2 c_j) = 0$. If $i \geq 1$, then $j \leq (s-1)/q^2 < 1 + \cdots + q^{2(l-1)}$, so

$$\deg_t \tau^2 c_j = \deg_t c_j \leq l - 1 \tag{12}$$

by induction hypothesis applied to j and $l - 1$ (note that $j < s$). Since the coefficients γ_i and δ_i do not depend on t, it follows from (11), (12) and (10) that $\deg_t c_s \leq l$ as required.

Let us now prove the second part of the lemma. We argue by induction on l. For $l = 0$ and $l = 1$ the assertion is true. Let now $l \geq 2$ be an integer, and suppose that the formula (9) holds for $l - 1$. Put $s := 1 + \cdots + q^{2(l-1)}$, and consider again the recursion formula (10). If j is any index appearing in the first sum, then, as before, $j < s = 1 + \cdots + q^{2(l-1)}$. Hence $\deg_t \tau c_j = \deg_t c_j \leq l - 1$ by the first part of the lemma. Let us now consider a pair (i, j) appearing in the second sum of (10). The smallest possible value for i is $i = 1$ (since $s \equiv 1 \pmod{q^2}$), for which we have $j = 1 + \cdots + q^{2(l-2)}$. In this case, the induction hypothesis yields (since $\delta_1 = -1$)

$$\delta_i(\tau^2 c_j) = (-1)^l t^{l-1} + \cdots$$

If now $i > 1$, then $j < 1 + \cdots + q^{2(l-2)}$, hence $\deg_t(\tau^2 c_j) = \deg_t c_j \leq l - 2$ by the first part of the lemma. It follows from these considerations that

$$(t - \theta^q) \sum_{i+jq^2=s} \delta_i(\tau^2 c_j) = (t - \theta^q)((-1)^l t^{l-1} + \cdots) = (-1)^l t^l + \cdots$$

Summing up, we have proved that $c_{1+q^2+\cdots+q^{2(l-1)}}(t) = (-1)^l t^l + \cdots$ $\square$

Remark. The following explicit formula can be deduced from (10), see [9].

$$\begin{aligned}\boldsymbol{d} =& 1 - (t-\theta)v - (t-\theta)v^{q^2-q+1} + (t-\theta)v^{q^2} + (t-\theta)(t - 2\theta^q + \theta)v^{q^2+1} \\ & - (t-\theta)(t-\theta^q)v^{q^2+q} + (\theta^q - \theta)(t-\theta)(t-\theta^q)v^{q^2+q+1} + \cdots \end{aligned} \tag{13}$$

3.3 Proof of Theorem 5

We can now begin the proof of Theorem 5. We write as before

$$\boldsymbol{d} = \sum_{s \geq 0} c_s v^s,$$

where $c_s \in A[t]$. It will be convenient to introduce the following notation. If $\boldsymbol{s} = (s_1, \ldots, s_k) \in \mathbb{N}^k$ (where $\mathbb{N} = \{0, 1, \ldots\}$), we define

$$||\boldsymbol{s}|| := \sum_{i=1}^{k} s_i q^{i-1}$$

and, as in Section 3.1, Equation (7),

$$d_{\boldsymbol{s}} := \det(\chi^{i-1}\tau^{j-1}c_{s_j})_{1 \leq i,j \leq k} = \det(C_{s_1}, C_{s_2}, \ldots, C_{s_k}),$$

where C_{s_j} is the column vector defined by

$$C_{s_j} = \begin{pmatrix} \tau^{j-1}c_{s_j} \\ \chi(\tau^{j-1}c_{s_j}) \\ \vdots \\ \chi^{k-1}(\tau^{j-1}c_{s_j}) \end{pmatrix}.$$

With this notation, the formula (6) writes

$$H_k(\boldsymbol{d}) = \sum_{\boldsymbol{s}} d_{\boldsymbol{s}} v^{||\boldsymbol{s}||}, \tag{14}$$

where $\boldsymbol{s}$ runs over all k-tuples of $\mathbb{N}^k$. To prove Theorem 5, we will show that the first non zero coefficient in the v-expansion (14) is obtained for only one multi-index $\boldsymbol{s}$, namely for

$$\boldsymbol{s}_0 := (1 + q^2 + \cdots + q^{2(k-2)}, \ldots, 1 + q^2, 1, 0).$$

This will easily yield the theorem. We will need for this three lemmas.

Lemma 10 *Set* $\boldsymbol{s}_0 := (1 + q^2 + \cdots + q^{2(k-2)}, \ldots, 1, 0) \in \mathbb{N}^k$. *Then we have*

$$||\boldsymbol{s}_0|| = \frac{(q^k-1)(q^{k-1}-1)}{(q^2-1)(q-1)}$$

and

$$d_{\boldsymbol{s}_0} = B_k(t).$$

Proof. The first part of the lemma amounts to compute the double sum

$$\sum_{i=1}^{k} \sum_{j=0}^{k-1-i} q^{2j} q^{i-1},$$

which is an exercise left to the reader.

To prove the second part, we use Lemma 9, Equality (9) :

$$d_{\boldsymbol{s}_0} = \begin{vmatrix} (-1)^{k-1}t^{k-1} + \cdots & \cdots & -t + \theta^{q^{k-2}} & 1 \\ (-1)^{k-1}t^{(k-1)q} + \cdots & \cdots & -t^q + \theta^{q^{k-2}} & 1 \\ \vdots & & \vdots & \vdots \\ (-1)^{k-1}t^{(k-1)q^{k-1}} + \cdots & \cdots & -t^{q^{k-1}} + \theta^{q^{k-2}} & 1 \end{vmatrix}.$$

Let us denote by $C_1, \ldots, C_k$ the columns of this matrix. If we substract $\theta^{q^{k-2}} C_k$ to C_{k-1}, then we eliminate the constant terms in C_{k-1}, that is, we get the new penultimate column $C'_{k-1} = {}^t(-t, -t^q, \ldots, -t^{q^{k-1}})$. By substracting now to the column C_{k-2} a suitable linear combination (with coefficients in $\mathbb{F}_q[\theta]$) of the last two columns, we get the new column $C''_{k-2} = {}^t(t^2, t^{2q}, \ldots, t^{2q^{k-1}})$. Repeating this process for the columns C_j, $j = k-3, \ldots, 1$, we see by induction that

$$d_{\boldsymbol{s}_0} = \begin{vmatrix} (-1)^{k-1}t^{k-1} & \cdots & -t & 1 \\ (-1)^{k-1}t^{(k-1)q} & \cdots & -t^q & 1 \\ \vdots & & \vdots & \vdots \\ (-1)^{k-1}t^{(k-1)q^{k-1}} & \cdots & -t^{q^{k-1}} & 1 \end{vmatrix}.$$

Now, this determinant is equal to

$$\begin{vmatrix} 1 & t & \cdots & t^{k-1} \\ 1 & t^q & \cdots & t^{(k-1)q} \\ \vdots & \vdots & & \vdots \\ 1 & t^{q^{k-1}} & \cdots & t^{(k-1)q^{k-1}} \end{vmatrix},$$

which is equal to $B_k(t)$ (Vandermonde determinant; see also Lemma 8). $\square$

The next lemma roughly says that if a coefficient $d_{\boldsymbol{s}}$ is not zero in (14), and if we reorder the coefficients c_{s_i} such that the sequence $(\deg_t c_{s_i})_i$ is non decreasing, then the degrees $\deg_t c_{s_i}$ grow at least linearly in i.

Lemma 11 *Let $\boldsymbol{d} = (s_1, \dots, s_k) \in \mathbb{N}^k$ such that $d_{\boldsymbol{s}} \neq 0$. Let $(i_1, \dots, i_k)$ be a permutation of the set $\{1, \dots, k\}$ such that*

$$\deg_t c_{s_{i_1}} \leq \cdots \leq \deg_t c_{s_{i_k}}.$$

Then, for all l, we have

$$\deg_t c_{s_{i_l}} \geq l - 1.$$

Proof. Let us write $d_{\boldsymbol{s}} = \det(C_{s_1}, \dots, C_{s_k})$. Suppose that there exists an l such that $\deg_t c_{s_{i_l}} \leq l-2$. Then, since the operator τ does not change the degree in t, the family $(c_{s_{i_1}}, \dots, \tau^{i_l - 1} c_{s_{i_l}})$ consists of l polynomials in $K[t]$ of degree $\leq l-2$, so they are linearly dependent over K. Hence there exist elements $\lambda_j \in K$, not all zero, such that

$$\sum_{j=1}^{l} \lambda_j \tau^{i_j - 1} c_{s_{i_j}} = 0.$$

If we now apply the operator χ^{i-1} $(1 \leq i \leq k)$, we find :

$$\sum_{j=1}^{l} \lambda_j \chi^{i-1} \tau^{i_j - 1} c_{s_{i_j}} = 0 \quad (1 \leq i \leq k).$$

In other words, we get $\sum_{j=1}^{l} \lambda_j C_{i_j} = 0$, that is, a non trivial linear combination of the columns $(i_1, \dots, i_l)$ in $d_{\boldsymbol{s}}$. Hence $d_{\boldsymbol{s}} = 0$, which is a contradiction. □

We introduce a further notation. If $\sigma \in S_{\{1,\dots,k\}}$ is a permutation of the set $\{1, \dots, k\}$ and if $\boldsymbol{s} = (s_1, \dots, s_k)$ is an element of $\mathbb{N}^k$, we define $\boldsymbol{s}^\sigma := (s_{\sigma(1)}, \dots, s_{\sigma(k)})$. We recall that $\boldsymbol{s}_0$ was defined in Lemma 10.

Lemma 12 *Let σ be a permutation of the set $\{1, \dots, k\}$ such that $\sigma \neq \mathrm{Id}$. Then*

$$||\boldsymbol{s}_0^\sigma|| > ||\boldsymbol{s}_0||.$$

Proof. We argue by induction on k. For $k = 1$ there is nothing to prove. Let now $k \geq 2$ be an integer and let σ be a permutation as in the lemma. For $l \geq 1$, define t_l by $t_l := 1 + \cdots + q^{2(l-2)}$. We will also use the notation $\boldsymbol{s}_0^{(k)}$ instead of $\boldsymbol{s}_0$ to indicate the dependence on k. Thus we have

$$\boldsymbol{s}_0 = \boldsymbol{s}_0^{(k)} = (t_k, \dots, t_1) \text{ and } ||\boldsymbol{s}_0^{(k)}|| = \sum_{l=1}^{k} t_l q^{k-l}.$$

Let further τ denote the permutation of $\{1, \dots, k\}$ such that $\big(\boldsymbol{s}_0^{(k)}\big)^\sigma = (t_{\tau(k)}, \dots, t_{\tau(1)})$.

First, suppose that $\tau(k) = k$. Then τ induces a non trivial permutation of the set $\{1, \dots, k-1\}$, and

$$||\big(\boldsymbol{s}_0^{(k)}\big)^\sigma|| - ||\boldsymbol{s}_0^{(k)}|| = \sum_{l=1}^{k-1} (t_{\tau(l)} - t_l) q^{k-l} = q\, ||\big(\boldsymbol{s}_0^{(k-1)}\big)^{\sigma'}|| - ||\boldsymbol{s}_0^{(k-1)}||,$$

where σ' is the (non trivial) permutation of $\{1, \dots, k-1\}$ such that $\big(\boldsymbol{s}_0^{(k-1)}\big)^{\sigma'} = (t_{\tau(k-1)}, \dots, t_{\tau(1)})$. By induction hypothesis, it immediately follows that $||\big(\boldsymbol{s}_0^{(k)}\big)^\sigma|| - ||\boldsymbol{s}_0^{(k)}|| > 0$.

Suppose now that $\tau(k) \neq k$. Then

$$||\big(\boldsymbol{s}_0^{(k)}\big)^\sigma|| \geq t_k q^{k-\tau^{-1}(k)} \geq q t_k = \frac{q(q^{2(k-1)}-1)}{q^2-1} > \frac{(q^k-1)(q^{k-1}-1)}{(q^2-1)(q-1)},$$

hence $||\big(\boldsymbol{s}_0^{(k)}\big)^\sigma > ||\boldsymbol{s}_0^{(k)}||$ by Lemma 10. $\square$

Proof of Theorem 5. We define $\boldsymbol{s}_0 = (s_{0,1}, \ldots, s_{0,k})$ as in Lemma 10. Thus we have

$$s_{0,l} = 1 + \cdots + q^{2(k-1-l)} \quad (1 \leq l \leq k).$$

Let now $\boldsymbol{s} = (s_1, \ldots, s_k) \in \mathbb{N}^k$ be such that $d_{\boldsymbol{s}} \neq 0$. Choose a permutation σ of $\{1, \ldots, k\}$ such that $\deg_t c_{s_{\sigma(k)}} \leq \cdots \leq \deg_t c_{s_{\sigma(1)}}$. By Lemma 11 (note the different order that we have chosen here), we have $\deg_t c_{s_{\sigma(l)}} \geq k - l$ for all l. Hence, by Lemma 9,

$$s_{\sigma(l)} \geq 1 + \cdots + q^{2(k-1-l)} = s_{0,l},$$

or

$$s_l \geq s_{0,\sigma^{-1}(l)}. \tag{15}$$

It follows, by Lemma 12, that we have

$$||\boldsymbol{s}|| \geq ||\boldsymbol{s}_0^{\sigma^{-1}}|| \geq ||\boldsymbol{s}_0||,$$

and the equality $||\boldsymbol{s}|| = ||\boldsymbol{s}_0||$ holds only if $\sigma = \mathrm{Id}$. In that case, the inequality (15) shows that $||\boldsymbol{s}|| = ||\boldsymbol{s}_0||$ only if $\boldsymbol{s} = \boldsymbol{s}_0$. Thus, we have shown that in the v-expansion (14), the first non zero coefficient is $d_{\boldsymbol{s}_0}$:

$$H_k(\boldsymbol{d}) = d_{\boldsymbol{s}_0} v^{||\boldsymbol{s}_0||} + \text{higher terms}$$

The points 1 and 2 of Theorem 5 follow at once from this and Lemma 10 (recall that $v = u^{q-1}$, so $\nu_k = (q-1)||\boldsymbol{s}_0||$). The point 3 is then a consequence of Proposition 7. $\square$

4 Proof of Theorem 1

In order to prove Theorem 1, we need to introduce a few notation. For any integer $l \geq 0$ and any triple $(\mu, \nu, m) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z}$, we denote by $\widetilde{\mathcal{M}}_{\mu,\nu,m}^{\leq l}$ the $K((t))$-module of almost A-quasimodular forms of weight (μ, ν), type m and depth $\leq l$ (see [9], Section 4.2), and we set

$$\widetilde{\mathcal{M}}_{\mu,\nu,m} = \bigcup_{l \geq 0} \widetilde{\mathcal{M}}_{\mu,\nu,m}^{\leq l}.$$

We will also write $l(f)$ for the depth of the form f. As in [9, § 5.1], we further set $\boldsymbol{h} = h\boldsymbol{d}$, and we finally define

$$\mathbb{M}^\sharp_{\mu,\nu,m} = K((t))[g, h, \Delta^{-1}, \boldsymbol{E}, \boldsymbol{h}] \cap \widetilde{\mathcal{M}}_{\mu,\nu,m}.$$

We have :

Lemma 13 *1. If $f \in \mathbb{M}^\sharp_{\mu,\nu,m}$, then $\tau f \in \mathbb{M}^\sharp_{q\mu,\nu,m}$ and $\chi f \in \mathbb{M}^\sharp_{\mu,q\nu,m}$.*

2. For all $j \geq 0$ we have

$$l(\tau^j \boldsymbol{E}) \leq 1, \quad l(\tau^j \boldsymbol{h}) \leq 1, \quad l(\chi^j \boldsymbol{E}) \leq q^j, \quad l(\chi^j \boldsymbol{h}) \leq q^j.$$

Proof. We first prove that the following equalities hold :

$$\tau \boldsymbol{h} = \Delta \boldsymbol{E}, \quad \tau \boldsymbol{E} = \frac{1}{t - \theta^q}(g\boldsymbol{E} + \boldsymbol{h}), \quad \chi \boldsymbol{h} = (t-\theta)^q \boldsymbol{E}^q - \frac{g}{\Delta}\boldsymbol{h}^q, \quad \chi \boldsymbol{E} = \frac{\boldsymbol{h}^q}{\Delta}. \tag{16}$$

The first equality follows at once from the definitions of $\boldsymbol{h}$ and $\boldsymbol{E}$ and the second is Lemma 22 of [9]. The last one then follows from the first :

$$\chi \boldsymbol{E} = \chi\left(\frac{\tau \boldsymbol{h}}{\Delta}\right) = \frac{\boldsymbol{h}^q}{\Delta}.$$

Finally, to prove the third equality, we use the following one, which follows for instance from [9, Proposition 9] or [4, Proposition 2.7] :

$$\boldsymbol{h} = \frac{t - \theta^q}{\Delta^q}(\tau^2 \boldsymbol{h}) - \frac{g}{\Delta}\tau \boldsymbol{h}.$$

Applying χ to both sides of this equality, and using the formula $\tau \boldsymbol{h} = \Delta \boldsymbol{E}$, we get

$$\chi \boldsymbol{h} = \frac{t^q - \theta^q}{\Delta^q}\tau(\boldsymbol{h}^q) - \frac{g}{\Delta}\boldsymbol{h}^q = (t-\theta)^q \boldsymbol{E}^q - \frac{g}{\Delta}\boldsymbol{h}^q.$$

The first part of the lemma follows at once from the relations (16) (we recall that $\boldsymbol{E} \in \widetilde{\mathcal{M}}^{\leq 1}_{1,1,1}$ and $\boldsymbol{h} \in \widetilde{\mathcal{M}}^{\leq 0}_{q,1,1}$). The second part is a simple induction, noticing that the depth of a form f in $\mathbb{M}^\sharp_{\mu,\nu,m}$ is nothing else than the degree $\deg_{\boldsymbol{E}} f$, when f is seen as an element of the polynomial ring $K((t))(g, h, \Delta^{-1})[\boldsymbol{E}, \boldsymbol{h}]$. □

We now have all the elements to prove Theorem 1.

Proof of Theorem 1. For all $i, j \in \{1, \ldots, k\}$ we have, by Lemma 13:

$$\chi^{i-1}\tau^{j-1}\boldsymbol{E} \in \widetilde{\mathcal{M}}^{\leq q^{i-1}}_{q^{j-1}, q^{i-1}, 1}.$$

It follows, by a straightforward computation, that

$$H_k(\boldsymbol{E}) \in \widetilde{\mathcal{M}}^{\leq (q^k-1)/(q-1)}_{(q^k-1)/(q-1), (q^k-1)/(q-1), k}. \tag{17}$$

Replacing t by θ, we then obtain the value of the weight and the type of $E_{j,k} = (\tau^j H_k(\boldsymbol{E})/B_k)|_{t=\theta}$. We prove the last part of the first property of the Theorem asserting that the degree in E of $E_{j,k}$ is not smaller than some integer l_k with $l_k \to \infty$ as $k \to \infty$.

By the main theorem of [9], if $f \in \widetilde{M}^{\leq l}_{w,m}$ is non-zero and if

$$w \geq 4l(2q(q+2)(3+2q)l + 3(q^2+1))^{3/2}, \tag{18}$$

then

$$\mathrm{ord}_{u=0} f \leq 16q^3(3+2q)^2 lw.$$

We can choose $C(q,k)$ big enough so that if $j \geq C(q,k)$, then (18) holds with $f = E_{j,k}$, $w = (q^k-1)(q^j+1)/(q-1)$ and $l = \deg_E(E_{j,k})$. Then, we get

$$l \geq \frac{q^{j-3}}{1+q^j}\frac{1}{16(1+q)(3+2q)^2}(1+q^k)$$

so that, enlarging $C(q,k)$ if necessary, we get, for $j \geq C(q,k)$,

$$l \geq \frac{1}{32(1+q)(3+2q)^2}(1+q^k)$$

which gives the required property of growth of the sequence $(l_k)_k$.

Using now Theorem 5 and (3), we find, for all $j \geq 0$:

$$\frac{\tau^j H_k(\boldsymbol{E})}{\kappa_{k,\nu_k}} = \frac{(-1)^k h^{q^j \frac{q^k-1}{q-1}}}{\kappa_{k,\nu_k}} \tau^{j+1}\left(\kappa_{k,\nu_k} u^{\nu_k} + \cdots\right) = (-1)^k h^{q^j \frac{q^k-1}{q-1}}(u^{q^{j+1}\nu_k} + \cdots) \in A[[t,u]].$$

Substituting $t = \theta$ in this equality yields

$$E_{j,k} = (-1)^k h^{q^j \frac{q^k-1}{q-1}}(u^{q^{j+1}\nu_k} + \cdots) \in A[[u]].$$

The properties 2, 3 of Theorem 1 follow at once from this and from (17).

It remains to show the property 4. We consider first the case $k = 1$. By definition, $H_1(\boldsymbol{E}) = \boldsymbol{E}$ and $E_{j,1} = (\tau^j \boldsymbol{E})|_{t=\theta}$ with $\mathrm{ord}_{u=0} E_{j,1} = q^j$. By [3, Theorem 1.2, Proposition 2.3], $E_{j,1}$ is proportional to the function x_j defined there, and hence extremal. Moreover, it is normalised, so that $E_{j,1} = f_{1,q^j+1,1}$ for $j \geq 0$.

Let us assume now that $k = 2, q \geq 3$. In this case, by [3, Theorem 1.3, Proposition 2.13], we see that $E_{j,2}$ is proportional to the form ξ_j defined there, and hence extremal. Since it is normalised and defined over A, the proof of Theorem 1 is complete. $\square$

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